# Landau-Lifshitz magnetodynamics as a Hamilton model: Magnons in an instanton background

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To take full advantage of the well-developed field-theoretic methods, Magnonics needs a yet-existing Lagrangian formulation. Here, we show that Landau-Lifshitz magnetodynamics is a member of the covariant-Schrödinger-equation family of Hamilton models and apply the covariant background method arriving at the Ginzburg-Landau Lagrangian formalism for magnons in an instanton background. Magnons appear to be nonrelativistic spinless bosons, which feel instantons as a gauge field and as a Bose condensate. Among the examples of the usefulness of the proposition is the recognition of the instanton-induced phase shifts in magnons as the Berry phase and the interpretation of the spin-transfer-torque generation as a ferromagnetic counterpart of the Josephson supercurrent.

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## I. INTRODUCTION

Nanometer-scale magnetization is believed to be one of the most promising fabrics for the computational platforms of the future. As long as the essence of computations is in the manipulation of the (nonlinear) correlations, it is the stable nonlinear magnetic entities—domain walls (DWs) (Ref. 1) or more exotic instantons[2,3]—which could serve as state variables. Magnons, in turn, are natural tools to transfer information within the system, to mediate the nonlocal exchange coupling, or to probe the magnetic configuration. From this perspective, the studies of the instanton-magnon interaction, or more generally of Magnonics,[4] are not only interesting fundamentally but also have the potential for the technological impact.

Even though the Landau-Lifshitz (LL) equation[5–10] is almost as old as the Schrödinger equation (SE), Magnonics is still relatively immature in comparison with the other field theories such as the nonrelativistic quantum mechanics itself (i.e., the SE). The reason is the absence of the action minimization principle (Lagrangian formulation, LF) underlying the dynamics of magnons. Only within the LF, the full potential of the well-developed field-theoretic methods could be applied to Magnonics. In particular, the dynamical Ginzburg-Landau (GL) diagrammatic technique for magnons will become available.

In this paper, we construct this necessary but missing tool for Magnonics. In Sec. II A, we derive the covariant GL LF for the LL magnetodynamics. In Secs. II B and II C, we show, using the theory of Kähler manifolds,[11] that the LL magnetodynamics belongs the family of Hamilton models, equations of motion of which are the covariant SEs (CSEs). In Sec. III, we use the covariant background field (CBF) decomposition method[12] to derive the GL action for magnons in a background of instantons. Section IV is devoted to the exemplification of the usefulness of our approach: we interpret the instanton-induced magnon phase shift[6] as the Berry phase (Sec. IV A); show that in addition to the Berry phase, instantons are also being felt by magnons as a Bose condensate (Sec. IV B); consider the case of a single DW (Ref. 9) to demonstrate the ease with which the proposed GL LF deals with the problem of magnon spectrum in a nontrivial instanton background (Sec. IV C); discuss the enabled dynamical diagrammatic technique (Sec. IV D), which can take a detailed account of the geometrical nonlinearities, which, in turn, are dominant for magnons as we show by considering the three-magnon scattering[10] in Sec. IV E; draw a parallel between the spin-transfer-torque (STT) generation and the Josephson phenomenon (Sec. IV F); and incorporate the soft-mode fluctuations into the proposed GL LF (Sec. IV G). Section V concludes our work.

## II. COVARIANT FORM OF MAGNETODYNAMICS

### A. Magnetodynamical Lagrangian formalism

According to the LL equation (LLE), the time evolution of the magnetization field, $\vec{m}(\boldsymbol{r},t)$, where $t$ is time and $\boldsymbol{r}=r^{\mu}$, $\mu=1,\ldots,D$ is a real-space point, is

$$\gamma^{-1}\partial_t\vec{m}=\vec{H}_{\text{eff}}\times\vec{m}. \tag{1}$$

Here $\partial_t=\partial/\partial t$, $\gamma$ is the gyromagnetic ratio, $\times$ denotes vector product, and $\vec{H}_{\text{eff}}(\boldsymbol{r},t)=\delta\mathcal{H}/\delta\vec{m}(\boldsymbol{r},t)$ is the effective magnetic field, given as a functional derivative of the GL energy functional

$$\mathcal{H}=\int_{\boldsymbol{r}}\mathcal{E},\mathcal{E}=\kappa(\partial_{\mu}\vec{m})^2+U, \tag{2}$$

where $\partial_{\mu}=\partial/\partial r^{\mu}$, $\kappa$ is the exchange constant, and $U$ is the potential, which in its "minimal" form is the sum of the anisotropy energy and the interaction with the external and the demagnetization magnetic fields.

We do not consider the energy dissipation issues, which makes Eq. (1) a conservative time evolution preserving the GL energy, $\mathcal{H}$. When needed, Gilbert damping,[13] or any other form of energy dissipation due to the coupling to an energy reservoir, can be incorporated into the LF either phenomenologically or systematically, e.g., within the Keldysh double-contour picture.[14]

In its conventional form (1), the magnetodynamics is in a sense overdetermined. It deals with $\vec{m}$ as though it has three

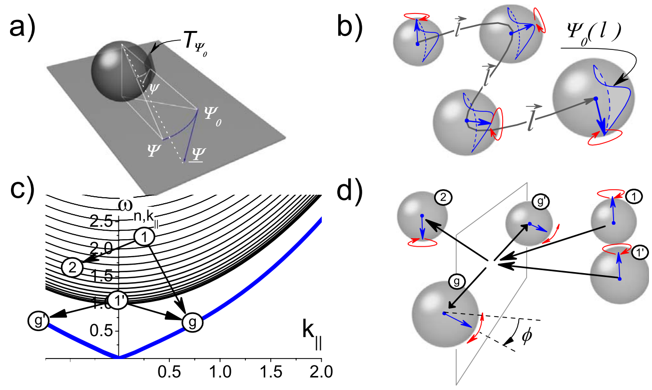


FIG. 1. (Color online) (a) The $\Psi$ coordinates on $S^2$ are obtained by the stereographic projection of $S^2$ onto the complex plane. $\Psi_0$ is the static magnetization representing instantons and $\Psi$ is the dynamical magnetization in the presence of magnonic excitations. $\underline{\Psi}$ (straight arrow) is the "geodesic" coordinate, which is the vector touching the geodesic connecting $\Psi_0$ and $\Psi$ and which belongs to the linear tangent space, $T_{\Psi_0}$. $\psi$ is the magnonic field, which is essentially $\underline{\Psi}$ in the basis of vielbeins. (b) As a magnon (circled arrows represent the magnonic gyration) propagates in space (path $\boldsymbol{l}$), it acquires the Berry phase, equal to the (oriented) area (light gray area) formed by $\Psi_0(\boldsymbol{l}) \in S^2$ and the geodesic connecting its initial and final positions (dashed curve). (c) In the presence of a DW, besides the bulk continuum (a set of thinner curves), the magnon spectrum has the GM. Also indicated the two possible DW-assisted three-magnon processes: the inelastic negative refraction of an incident bulk magnon into another bulk magnon with the accompanying creation of a DW-GM boson $(1)\to(2)+(g)$; and the absorption of an incident bulk magnon by the DW with the creation of two DW-GM bosons: $(1')\to(g)+(g')$. (d) Graphical representation of the two DW-assisted three-magnon processes. $\phi$ is the azimuthal angle of the magnetization on the DW, fluctuation in which is the essence of the DW-GM.

independent components, whereas Eq. (1) (even in the presence of Gilbert damping) preserves $|\vec{m}|=m_0$ so that $\vec{m}$ lives on a two-dimensional (2D) sphere, $S^2$, which is also known as the projective complex line $CP^1 \simeq S^2$. Therefore, one can introduce the 2D coordinates on $S^2$ by the stereographic projection onto a complex plane, $\mathcal{C}^1$ [see Fig. 1(a)]

$$m^+ = m_x + im_y = 2m_0 e^{-\mathcal{K}}\Psi, \quad m_z = m_0 e^{-\mathcal{K}}(1-\Psi\bar{\Psi}),$$

where $\mathcal{K}=\ln(1+\Psi\bar{\Psi})$ is known as the Kähler potential on $S^2$ (see below). The induced (Fubini-Study) metric on $S^2$ is

$$(d\vec{m})^2 = 2m_0^2 g_{ab} d\Psi^a d\Psi^b, \quad g_{ab} = \begin{pmatrix} 0 & g_{1\bar{1}} \\ g_{\bar{1}1} & 0 \end{pmatrix}, \tag{3}$$

where $a,b=(1,\bar{1})$, $\Psi^1\equiv\Psi$, $\Psi^{\bar{1}}\equiv\bar{\Psi}$, and $\Psi^a\equiv(\Psi,\bar{\Psi})$, the Einstein's summation over the repeated indices is assumed throughout the paper, and $g_{1\bar{1}}=g_{\bar{1}1}=\partial_1\partial_{\bar{1}}\mathcal{K}=e^{-2\mathcal{K}}$ with $\partial_a \equiv \partial/\partial\Psi^a$.[15]

To obtain the LLE in terms of the new fields, we first write down the equation for $m^+$,

$$i\mu_t e^{-2\mathcal{K}}(\partial_t\Psi - \Psi^2\partial_t\bar{\Psi}) = -\mu_r^2 e^{-2\mathcal{K}}(\mathcal{D}_\mu\partial_\mu\Psi + \Psi^2\mathcal{D}_\mu\partial_\mu\bar{\Psi}) + \tilde{U}, \tag{4a}$$

where $\mathcal{D}_\mu\partial_\mu\Psi=(\partial_\mu+\Gamma^1_{11}\partial_\mu\Psi)\partial_\mu\Psi$ and $\mathcal{D}_\mu\partial_\mu\bar{\Psi}=(\partial_\mu+\Gamma^{\bar{1}}_{\bar{1}\bar{1}}\partial_\mu\bar{\Psi})\partial_\mu\bar{\Psi}$, with $\Gamma^1_{11}=-2e^{-\mathcal{K}}\bar{\Psi}$ and $\Gamma^{\bar{1}}_{\bar{1}\bar{1}}=-2e^{-\mathcal{K}}\bar{\Psi}$ being the Christoffel symbols (see below), $\mu_t=2m_0\gamma^{-1}$, $\mu_r^2=4m_0^2\kappa$, and the potential term is

$$\tilde{U} = \left[ m_z\left(\frac{\partial\Psi^a}{\partial m_x} + i\frac{\partial\Psi^a}{\partial m_y}\right) - m^+\frac{\partial\Psi^a}{\partial m_z}\right]\delta_{\Psi^a}U,$$

where $\delta_{\Psi^a}\equiv\delta/\delta\Psi^a(\boldsymbol{r},t)$. Further, we recall that vector $\vec{m}$ is subject to the constraint $|\vec{m}|=0$ and there are only two independent components, say $m_x$ and $m_y$.[16] Therefore, one can assume that $\Psi^a(\vec{m})\to\Psi^a(m_x,m_y)$ so that the last term in the definition of $\tilde{U}$ vanishes. The remaining part can be elaborated by the following Jacobian of the coordinate transformation:

$$\frac{\partial\Psi^a}{\partial m_j} = \frac{1}{2m_z}\begin{bmatrix} 1+\Psi^2 & 1+\bar{\Psi}^2 \\ i(1-\Psi^2) & -i(1-\bar{\Psi}^2) \end{bmatrix}_{ja}, \quad j=x,y,$$

leading to

$$\tilde{U} = \delta_{\bar{\Psi}}U + \Psi^2\delta_\Psi U. \tag{4b}$$

Now, we take the equation, which is a complex conjugate of Eq. (4), multiply it by $\Psi^2$, subtract it from Eq. (4) itself, rescale the coordinates as $t\to\mu_t t$, $\boldsymbol{r}\to\mu_r\boldsymbol{r}$, and drop out the common factor $(1-|\Psi|^4)e^{-2\mathcal{K}}$. The result is the following $(g^{ac}g_{cb}=\delta^a_b)$:

$$i\partial_t\Psi = -\mathcal{D}_\mu\partial_\mu\Psi + g^{1a}\delta_{\Psi^a}U. \tag{5a}$$

Combining Eq. (5a) with the similar equation for the antiholomorphic field, $\bar{\Psi}$, we arrive at

$$\mathcal{J}^a_b\partial_t\Psi^b = -\mathcal{D}_\mu\partial_\mu\Psi^a + g^{ab}\delta_{\Psi^b}U, \tag{5b}$$

where $\mathcal{J}^b_a=\text{diag}(i,-i)$ is the so-called complex structure (see below). After lowering indices in Eq. (5b) by $g_{ab}$ we get

$$\Omega_{ab}\partial_t\Psi^b = -g_{ab}\mathcal{D}_\mu\partial_\mu\Psi^b + \delta_{\Psi^a}U, \tag{5c}$$

where $\Omega_{ab}=g_{ac}\mathcal{J}^c_b$ is the so-called Kähler form (see below).

As can be straightforwardly verified, Eq. (5) is the equation of motion (EoM) of the action, $\mathcal{S}=\int_{t,\boldsymbol{r}}\mathcal{L}$, where the Lagrangian density is

$$\mathcal{L} = \theta_a\partial_t\Psi^a - (g_{ab}\partial_\mu\Psi^a\partial_\mu\Psi^b/2 + U) \tag{6a}$$

with $\theta_1=\overline{\theta_{\bar{1}}}=(i/2)\partial_1\mathcal{K}=(i/2)e^{-\mathcal{K}}\bar{\Psi}$ or

$$\theta_a = \mathcal{J}^b_a\partial_b\mathcal{K}/2, \tag{6b}$$

which is known as the Poincáre 1-form–a vector field on $S^2$ obeying $\partial_a\theta_b - \partial_b\theta_a = \Omega_{ab}$ (see below).

The GL part of Eq. (6a) (the one in the parenthesis) is merely Eq. (2) in terms of $\Psi$'s, whereas it can be shown that the dynamical part (the first term) is equivalent to $\vec{A}\cdot\partial_t\vec{m}$,[5] where $\vec{A}$ is the vector potential of a Dirac string.[17] Thus, Eq.

(6) is equivalent to the standard LF for magnetodynamics used, e.g., in Ref. 5. The advantage of our formulation is solely in the covariance, which cuts deeper than it may seem from the first sight. In particular, it is the covariance of Eq. (6) which will allow to derive the magnonic action to all the orders of the magnon-magnon and instanton-magnon interactions in Sec. III below.

### B. Covariant-Schrödinger-equation family of Hamilton models

The above Lagrangian formalism for the LL magnetodynamics actually belongs to the entire family of Hamilton models for fields, which may live on any Kähler manifold, $\mathcal{M}$, not only $S^2$. The construction of this family of models will allow firstly to clarify the geometrical meaning of the objects, introduced in the previous section, and secondary to establish the covariance of Eq. (6), which is needed to justify the application of the CBF method for the derivation of the magnonic action in Sec. III below. For this reason, let us digress here on this generalization of the LL magnetodynamics and on the brief review of the theory of $\mathcal{M}$'s.[11]

Any $\mathcal{M}$ has even number of real dimensions, $\dim_{\mathbb{R}} \mathcal{M} = 2d$, where $d = \dim_{\mathbb{C}} \mathcal{M}$ is its complex dimensionality. The coordinates on $\mathcal{M}$ can be split into the holomorphic and antiholomorphic sets: $\Psi^{\eta}$ and $\bar{\Psi}^{\bar{\eta}}$, $\eta$, $\bar{\eta}=1,\ldots,d$. The sets are actually complex conjugates of each other ($\overline{\Psi^{\eta}}=\bar{\Psi}^{\bar{\eta}}$). However, it is convenient to treat them as independent and introduce the total Latin indices $\Psi^a \equiv (\Psi^{\eta}, \bar{\Psi}^{\bar{\eta}})$, i.e., $a=\eta \oplus \bar{\eta}$, as opposed to the Greek indices, which run over only either holomorphic or antiholomorphic components. The two sets are geometrically separated by the so-called complex structure ($a=\eta\oplus\bar{\eta}$, $b=\nu\oplus\bar{\nu}$),

$$\mathcal{J}^a_b = i\begin{pmatrix} 1^{\eta}_{\nu} & 0^{\eta}_{\bar{\nu}} \\ 0^{\bar{\eta}}_{\nu} & -1^{\bar{\eta}}_{\bar{\nu}} \end{pmatrix}, \tag{7}$$

existence of which makes $\mathcal{M}$ a complex manifold.[18]

Any $\mathcal{M}$ is Riemannian, i.e., it is provided with the concept of the infinitesimal length, $dl^2 = g^{\mathcal{M}}_{ab} d\Psi^a \otimes d\Psi^b$. The metric tensor is ($a=\eta\oplus\bar{\eta}$, $b=\nu\oplus\bar{\nu}$),

$$g^{\mathcal{M}}_{ab} = \begin{pmatrix} 0_{\eta\nu} & g^{\mathcal{M}}_{\eta\bar{\nu}} \\ g^{\mathcal{M}}_{\bar{\eta}\nu} & 0_{\bar{\eta}\bar{\nu}} \end{pmatrix}, \tag{8}$$

where the nonzero components can be given via the Kähler potential: $g^{\mathcal{M}}_{\eta\bar{\nu}} = \overline{g^{\mathcal{M}}_{\bar{\eta}\nu}} = \partial_{\eta}\partial_{\bar{\nu}}\mathcal{K}^{\mathcal{M}}$, with $\partial_{\eta}=\partial/\partial\Psi^{\eta}$, $\partial_{\bar{\eta}}=\partial/\partial\bar{\Psi}^{\bar{\eta}}$.

Let us recall now that the Hamilton dynamics can be defined only on a symplectic manifold, $\mathcal{X}$, i.e., the one which admits a closed nondegenerate (symplectic) 2-form, $\Omega^{\mathcal{X}}_{ab} = -\Omega^{\mathcal{X}}_{ba}$ and $\mathbf{d}\Omega^{\mathcal{X}} \equiv \partial_{[a}\Omega^{\mathcal{X}}_{bc]}=0$, where square brackets denote antisymmetrization and $\mathbf{d}$ is known as the exterior derivative. The Hamilton EoMs for the fields, $X^a(\boldsymbol{r},t)\in\mathcal{X}$, is

$$\Omega^{\mathcal{M}}_{ab}\partial_t X^b = \delta\mathcal{H}^{\mathcal{X}}/\delta X^a, \tag{9}$$

which corresponds to the following action $\mathcal{S}^{\mathcal{X}} = \int_t [(\int_{\boldsymbol{r}} \mathcal{T}^{\mathcal{X}}) - \mathcal{H}^{\mathcal{X}}]$, where $\mathcal{H}^{\mathcal{X}}$ is some energy functional of $X$'s and $\mathcal{T}^{\mathcal{X}} = \theta^{\mathcal{X}}_a \partial_t X^a$ with $\theta^{\mathcal{X}}_a$ such that $\Omega^{\mathcal{X}}_{ab} = \mathbf{d}\theta^{\mathcal{X}} \equiv \partial_{[a}\theta^{\mathcal{X}}_{b]}$. The (local) existence of $\theta^{\mathcal{X}}$ is assured by the Poincaré lemma, which states that if a form is closed, i.e., $\mathbf{d}\Omega^{\mathcal{X}}=0$, it is also (locally) exact, i.e., $\Omega^{\mathcal{X}} = \mathbf{d}\theta^{\mathcal{X}}$.

Fortunately, all $\mathcal{M}$'s are symplectic with $\Omega$ being the so-called Kähler form, $\Omega^{\mathcal{M}}_{ab} = g^{\mathcal{M}}_{ac}\mathcal{J}^c_b$. As can be straightforwardly verified, $\Omega^{\mathcal{M}} = \mathbf{d}\theta^{\mathcal{M}}$ with

$$\theta^{\mathcal{M}}_a = \mathcal{J}^b_a \partial_b \mathcal{K}^{\mathcal{M}}/2. \tag{10a}$$

Thus, the Hamilton-dynamics-defining part of the Lagrangian density is $\mathcal{T}^{\mathcal{M}} = \theta^{\mathcal{M}}_a \partial_t \Psi^a$, which is holomorphically covariant, i.e., it preserves its form under the coordinate transformations, which do not mix $\Psi$'s and $\bar{\Psi}$'s and thus leave the complex structure unchanged.

Now we can construct the action $\mathcal{S}^{\mathcal{M}} = \int_{\boldsymbol{r},t}\mathcal{L}^{\mathcal{M}}$, $\mathcal{L}^{\mathcal{M}} = \mathcal{T}^{\mathcal{M}} - \mathcal{E}^{\mathcal{M}}$, where the last term has the standard nonlinear sigma model (NLSM) form, $\mathcal{E}^{\mathcal{M}} = g^{\mathcal{M}}_{ab}\partial_{\mu}\Psi^a\partial_{\mu}\Psi^b/2 + U$, so that

$$\mathcal{L}^{\mathcal{M}} = \theta^{\mathcal{M}}_a \partial_t \Psi^a - (g^{\mathcal{M}}_{ab}\partial_{\mu}\Psi^a\partial_{\mu}\Psi^b/2 + U). \tag{10b}$$

Note that it is the simple relation between $g^{\mathcal{M}}$ and $\Omega^{\mathcal{M}}$, which is unique for $\mathcal{M}$s and which assures the (holomorphic) covariance of both $\mathcal{T}^{\mathcal{M}}$ and $\mathcal{E}^{\mathcal{M}}$.

The EoMs [Eq. (9)] now read

$$\Omega^{\mathcal{M}}_{ab}\partial_t \Psi^b = -g^{\mathcal{M}}_{ab}\mathcal{D}_{\mu}\partial_{\mu}\Psi^b + \delta_{\Psi^a}U, \tag{11a}$$

where $\mathcal{D}_{\mu}\partial_{\mu}\Psi^a = (\delta^a_b\partial_{\mu} + \Gamma^a_{bc}\partial_{\mu}\Psi^c)\partial_{\mu}\Psi^b$ is the Laplacian with the only nonzero Christoffel symbols being $\Gamma^{\eta}_{\nu\xi} = \overline{\Gamma^{\bar{\eta}}_{\bar{\nu}\bar{\xi}}} = (g^{\mathcal{M}})^{\eta\bar{\eta}}\partial_{\nu}g^{\mathcal{M}}_{\bar{\eta}\xi}$ $[(g^{\mathcal{M}})^{ac}g^{\mathcal{M}}_{cb} = \delta^a_b]$.[19] After rising indices in Eq. (11a) by $(g^{\mathcal{M}})^{ab}$, we get

$$\mathcal{J}^a_b\partial_t\Psi^b = -\mathcal{D}_{\mu}\partial_{\mu}\Psi^a + (g^{\mathcal{M}})^{ab}\delta_{\Psi^b}U, \tag{11b}$$

which for only holomorphic fields reduces to

$$i\partial_t\Psi^{\eta} = -\mathcal{D}_{\mu}\partial_{\mu}\Psi^{\eta} + (g^{\mathcal{M}})^{\eta\bar{\nu}}\delta_{\bar{\Psi}^{\bar{\nu}}}U. \tag{11c}$$

Equations (10) and (11) are arranged in the exactly opposite order than that of Eqs. (5) and (6). This is meant to emphasize that unlike in the previous section, where the LF for the magnetodynamics was derived starting from the phenomenological LLE [Eq. (1)], the above construction of the general-$\mathcal{M}$ LF is based on the covariance arguments and the theory of $\mathcal{M}$'s.

### C. Landau-Lifshitz equation as the covariant-Schrödinger equation

Equations (10) and (11) generalize the magnetodynamics defined by Eqs. (5) and (6) and represent the entire family of Hamilton models. Each member of this family is uniquely defined by the Kähler target space, $\mathcal{M}$, and the dimensionality of the base (real) space. The "seed" representative is the nonrelativistic quantum mechanics with $\Psi\in\mathcal{C}^1$ and $\mathcal{K}^{\mathcal{C}^1} = \Psi\bar{\Psi}$ so that $g^{\mathcal{C}^1}_{1\bar{1}}=1$, $\mathcal{E}^{\mathcal{C}^1} = \partial_{\mu}\Psi\partial_{\mu}\bar{\Psi} + U$ is the Hamiltonian, $\mathcal{T}^{\mathcal{C}^1} = (i/2)(\bar{\Psi}\partial_t\Psi - \text{c.c.})$, and the EoM is the conventional SE,

$$i\partial_t\Psi = -\partial^2_{\mu}\Psi + \delta_{\bar{\Psi}}U.$$

From this perspective, the generic-$\mathcal{M}$ EoMs, together with the LLE, can be called the CSEs. It is worth mentioning also that unlike for the SE, the nonlinearities enter into the CSEs not only through the potential, $U$, but also through the non-vanishing $\Gamma$'s.

The mere identification of the LLE as the CSE is interesting aesthetically but does not explicitly lead to new physical results. However, it can be used for mapping some yet-to-be-solved problems of magnetodynamics onto problems from other areas of physics, which have longer history and are better understood. For example, the problem of the phase locking between STT generators,[20] formulated in terms of the "complex order parameter," $\Psi$, seems similar in spirit to the Josephson phenomenon.[21] In Sec. IV F, we will apply this picture to the problem of a single STT generator.

## III. LAGRANGIAN FORMALISM FOR MAGNONS IN AN INSTANTON BACKGROUND

### A. Covariant background field method

Turning now to the construction of the Lagrangian formalism for magnons, we note that magnons are the dynamical fluctuations of the magnetization around its stationary configurations, $\Psi_0(\boldsymbol{r})$, which is a "local" minimum of $\mathcal{H}$ and which represents one of the infinite number of possible instanton backgrounds, i.e., "magnon vacua." The naive difference, $\delta\Psi(\boldsymbol{r},t)=\Psi(\boldsymbol{r},t)-\Psi_0(\boldsymbol{r})$, however, is a bad choice for the magnon field. Indeed, $S^2$ is not a linear space and $\delta\Psi$ is ill defined from the geometrical point of view. From the physical point of view, magnons must exhibit such effects as interference, i.e., the magnon fields should observe the principle of superposition and thus must belong to a linear space, while $\delta\Psi$'s do not. The linear space under consideration is the tangent space, $T_{\Psi_0(\boldsymbol{r})}\simeq\mathcal{C}^1$—a complex plane touching $S^2$ at point $\Psi_0(\boldsymbol{r})\in S^2$ [see Fig. 1(a)].

The appropriate way to introduce the magnon field is known as the CBF method, which is a basic ingredient of all NLSMs.[12] What is new in our case, compared with the conventional NLSMs, is the dynamical term in the Lagrangian, $\mathcal{T}$, which is only first order in the time derivative of the fields. However, by the virtue of the covariance of the magnetodynamical action [Eq. (6)], this does not introduce any additional complications and the CBF method is directly applicable in our case.

One starts with defining the one-parameter geodesic flow, $\Upsilon(\boldsymbol{r},t,s)$, $s\in[0,1]$, which connects the background configuration, $\Upsilon(\boldsymbol{r},t,0)=\Psi_0(\boldsymbol{r})$, and the magnonically excited configuration, $\Upsilon(\boldsymbol{r},t,1)=\Psi(\boldsymbol{r},t)$, and satisfies

$$\partial_s\underline{\Upsilon}+\Gamma^1_{11}(\Upsilon)\underline{\Upsilon}\underline{\Upsilon}=0, \tag{12}$$

where $\underline{\Upsilon}(\boldsymbol{r},t,s)\equiv\partial_s\Upsilon(\boldsymbol{r},t,s)$. Now, according to the physical essence of magnons as of fluctuations, the magnonic action is $\Delta\mathcal{S}\equiv\mathcal{S}(\Psi)-\mathcal{S}(\Psi_0)=\mathcal{S}(\Upsilon)|_{s=0}^{s=1}$, with $\mathcal{S}$ from Eq. (6). It can be given as a Taylor series,

$$\Delta\mathcal{S}=\sum\nolimits_{n=1}^{\infty}(1/n!)\mathcal{S}^{(n)}(\Upsilon)\big|_{s=0}, \tag{13a}$$

$$\mathcal{S}^{(n)}=(\partial_s)^n\mathcal{S}(\Upsilon)=\int_{\boldsymbol{r},t}\mathcal{L}^{(n)}(\Upsilon). \tag{13b}$$

The first $s$ derivative of the action results,[22]

$$\mathcal{L}^{(1)}=\underline{\Upsilon}^a(g_{ab}\mathcal{J}^b_c\partial_t\Upsilon^c+g_{ab}\mathcal{D}_\mu\partial_\mu\Upsilon^b-\partial_aU). \tag{14a}$$

To simplify matters, we consider here only the case of the local potential, though the generalization to the nonlocal $U$ is possible.

The next task is to find $\mathcal{L}^{(n)}$ to a desired order using the recursion: $\mathcal{S}^{(n)}=\partial_s\mathcal{S}^{(n-1)}$. It is easier, however, to introduce the covariant differentiation operator, $D_s$,[12] which has the same effect on the action: $\partial_s^n\mathcal{S}^{(1)}=D_s^n\mathcal{S}^{(1)}$. The introduction of $D_s$ is a convenient way to use Eq. (12) and express all the higher order $s$ derivatives, $\partial_s^n\Upsilon$, $n>1$, in terms of only $\Upsilon$ and $\underline{\Upsilon}$. $D_s$ acts on tensors, which are functions of $\Upsilon$ only, as $D_sA^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_m}=A^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_m|\gamma}\underline{\Upsilon}^\gamma$, where $|$ denotes the covariant derivative on $S^2$. In particular, $D_sg_{ab}=g_{ab|c}\underline{\Upsilon}^c=0$. Furthermore, $D_s$ acts on terms which involve $\underline{\Upsilon}^a$ as

$$D_s\underline{\Upsilon}^a=0,$$

$$D_x\underline{\Upsilon}^a\equiv D_s\partial_x\Upsilon^a=\partial_x\underline{\Upsilon}^a+\Gamma^a_{bc}(\partial_x\Upsilon^b)\underline{\Upsilon}^c,$$

$$D_sD_x\underline{\Upsilon}^a=\mathcal{R}^a_{bcd}\underline{\Upsilon}^b\underline{\Upsilon}^c(\partial_x\Upsilon^d)=-\mathcal{J}^a_b\underline{\Upsilon}^bJ_x,$$

$$D_s^2D_x\underline{\Upsilon}^a=\mathcal{R}^a_{bcd}\underline{\Upsilon}^b\underline{\Upsilon}^c(D_x\underline{\Upsilon}^d)=-\mathcal{J}^a_b\underline{\Upsilon}^bj_x,$$

where the subscript $x$ can be either $\mu$ or $t$, $\mathcal{R}_{abcd}=g_{ad}g_{bc}-g_{ac}g_{bd}$ is the Riemann curvature tensor on $S^2$, and we introduced

$$J_x=\underline{\Upsilon}^ag_{ab}\mathcal{J}^b_c\partial_x\Upsilon^c \quad \text{and} \quad j_x=\underline{\Upsilon}^bg_{ab}\mathcal{J}^b_cD_x\underline{\Upsilon}^c.$$

Note that $D_s\mathcal{R}_{abcd}=0$ as it should for the constant curvature $S^2$. Other useful relations needed in the following are $D_sJ_x=j_x$, $D_sj_x=J_x\omega$, and consequently

$$D_s^{2k+1}D_x\underline{\Upsilon}^a=-\mathcal{J}^a_b\underline{\Upsilon}^b\omega^kJ_x,$$

$$D_s^{2k+2}D_x\underline{\Upsilon}^a=-\mathcal{J}^a_b\underline{\Upsilon}^b\omega^kj_x,$$

where we introduced $\omega=\underline{\Upsilon}^ag_{ab}\underline{\Upsilon}^b$ (note that $D_s\omega=0$).

In the new notations, Eq. (14a) can be rewritten as

$$\mathcal{L}^{(1)}=J_t-(D_\mu\underline{\Upsilon}^a\partial_\mu\Upsilon_a+U_{|a}\underline{\Upsilon}^a). \tag{14b}$$

Acting once by $D_s$ on Eq. (14b) we arrive at the second (Gaussian) term,

$$\mathcal{L}^{(2)}=j_t-(D_\mu\underline{\Upsilon}^aD_\mu\underline{\Upsilon}_a+J_\mu^2+U_{|ab}\underline{\Upsilon}^a\underline{\Upsilon}^b). \tag{15}$$

Further acting recursively by $D_s$ on $\mathcal{S}^{(2)}$, higher order terms in Eq. (13) turn out to be

$$\mathcal{L}^{(n)}=\begin{cases}\omega^{k-1}(\omega J_t-4^kJ_\mu j_\mu)-U^{(n)},\\ \omega^{k-1}[\omega j_t-4^k(j_\mu^2+\omega J_\mu^2)]-U^{(n)},\end{cases} \tag{16}$$

where the first line is for odd $n=2k+1$, the second line is for even $n=2k+2$, and $U^{(n)}=U_{|a_1\cdots a_n}\underline{\Upsilon}^{a_1}\cdots\underline{\Upsilon}^{a_n}$. To get the magnonic action out of Eqs. (13a), (13b), (14a), (14b), (15), and (16), one allows $s\to 0$ by the following substitutions:

$$\Upsilon(\boldsymbol{r},t,s)\to\Upsilon(\boldsymbol{r},t,0)\equiv\Psi_0(\boldsymbol{r},t),$$

$$\underline{\Upsilon}(\boldsymbol{r},t,s)\to\underline{\Upsilon}(\boldsymbol{r},t,0)\equiv\underline{\Psi}(\boldsymbol{r},t),$$

and

$$\mathcal{L}^{(n)}[\Upsilon(\boldsymbol{r},t,s)] \to \mathcal{L}^{(n)}[\Psi_0(\boldsymbol{r},t),\underline{\Psi}(\boldsymbol{r},t)].$$

Note also that the term in the parentheses in Eq. (14a) is zero on $s\to 0$ due to the EoM [Eq. (5c)]. Therefore, $\mathcal{L}^{(1)}$ does not contribute to the magnonic action [Eq. (13)], which thus starts with the second (Gaussian) term, as it should for a stable instanton background/magnon vacuum.

### B. Magnon field: Vielbeins

The predecessor of the magnon field is $\underline{\Psi}(\boldsymbol{r},t)\in T_{\Psi_0(\boldsymbol{r})}$. It is the vector field touching $S^2$ along the geodesic, $\Upsilon(\boldsymbol{r},t,s)$, connecting the background magnetization configuration, $\Psi_0(\boldsymbol{r})$, and the magnonically excited configuration, $\Psi(\boldsymbol{r},t)$. The magnitude of $\underline{\Psi}(\boldsymbol{r},t)$ related to the "length" of this geodesic. The metric for $\underline{\Psi}(\boldsymbol{r},t)$, which for now is $g_{1\bar{1}}[\Psi_0(\boldsymbol{r})]$, is very inconveniently dependent on the position on $S^2$, whereas $S^2$ is a homogeneous space with all the points being equivalent to each other.

By the virtue of that $T_{\Psi_0(\boldsymbol{r})}$ is a linear space, one has the freedom to chose on it any linear basis. The basis has to be chosen from the requirement that the metric on $T_{\Psi_0(\boldsymbol{r})}$ is independent on $\Psi_0(\boldsymbol{r})$. Such basis is known as vielbeins in which

$$\underline{\Psi}^a(\boldsymbol{r},t) \to \psi^a(\boldsymbol{r},t) = e^{-\mathcal{K}_0}\underline{\Psi}^a(\boldsymbol{r},t), \tag{17a}$$

$$g_{ab} \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{ab}, \tag{17b}$$

$$D_x\underline{\Psi}^a \to \nabla_x\psi^a = (\partial_x \pm iA_x)\psi^a, \tag{17c}$$

$$J_x \to \bar{\psi}e^{-\mathcal{K}_0}i\partial_x\Psi_0 + \text{c.c.}, \tag{17d}$$

$$j_x \to \bar{\psi}i\nabla_x\psi + \text{c.c.}, \tag{17e}$$

$$\omega \to 2|\psi|^2, \tag{17f}$$

$$U^{(n)} \to e^{n\mathcal{K}_0}U_{|a_1\cdots a_n}\psi^{a_1}\cdots\psi^{a_n}, \tag{17g}$$

where $x$ again denotes either $\mu$ or $t$, $\mathcal{K}_0\equiv\mathcal{K}[\Psi_0(\boldsymbol{r})]$, $\pm$ is for $a=1,\ \bar{1}$, respectively, and the gauge field is

$$A_x = e^{-\mathcal{K}_0}\bar{\Psi}_0(i\partial_x\Psi_0) + \text{c.c.} = 2\theta_a\partial_x\Psi_0^a, \tag{18}$$

with $\theta_a$ being the Poincáre 1-form from Eq. (6b). Note also that $A_t,\ J_t=0$ as long as $\partial_t\Psi_0=0$.

$\Psi(\boldsymbol{r},t)$ and $\Psi_0(\boldsymbol{r})$ uniquely define the configuration of the magnon field, $\psi(\boldsymbol{r},t)\in T_{\Psi_0(\boldsymbol{r})}$, which thus can be used instead of $\delta\Psi(\boldsymbol{r},t)$ as discussed in the beginning of Sec. II A.

### C. Ginzburg-Landau magnonic action

In terms of $\psi$'s, the magnonic action is $\Delta\mathcal{S}=\int_{r,t}\Delta\mathcal{L}$, with the Lagrangian density (up to the third order explicitly) being

$$\Delta\mathcal{L} = \bar{\psi}i\nabla_t\psi - (|\nabla_\mu\psi|^2 + J_\mu^2/2 + U^{(2)}/2) + (2|\psi|^2J_t - 4J_\mu j_\mu - U^{(3)})/6 + \sum_{n=4}^{\infty}\mathcal{L}^{(n)}/n!, \tag{19}$$

where $\mathcal{L}^{(n)}$'s ($n>3$) are given by Eq. (16) with the appropriate substitutions of Eq. (17).

## IV. DISCUSSION

Previous sections contain the main results of this work, while here we would like to exemplify the advantages of our approach, to discuss some new insights enabled by it, and to outline possible future developments. Before we proceed several comments are in order. Throughout the paper, we kept referring to the two LFs given by Eq. (6) and Eq. (19) as the LL magnetodynamics and Magnonics, respectively. It is understood, however, that the two LFs are in fact equivalent to each other. The question of which of the two to use is a matter of convenience. The use of Magnonic LF (19) is preferable for those problems, which deal with relatively small fluctuations in the magnetization (Secs. IV A–IV E), while for the "full-range-magnetization" problems the magnetodynamical LF (19) is convenient (Secs. IV F and IV G).

Equation (19) is the Lagrangian density of the nonrelativistic spinless bosons, whereas magnons, being the excitation of the 3D-vector-Magnetization, are typically referred to as a vector (spin-1) excitation. The point here is that in the fixed-magnitude approsimation, which we have used so far (see, however, Sec. IV G for the generalization), the target space is 2D and the magnon field has only two real components, which being combined into a complex "boson wave function," $\psi$, leave no other indices for the field, i.e., the zero-spin complex field.

### A. Berry phase

The above analogy between quantum particles and magnons allows to draw parallels between quantum mechanics and Magnonics. Indeed, it is a straightforward observation that the coupling to the gauge field $A_\mu$ (via the "long" derivatives $\nabla_\mu$) results in the instanton-induced phase shift in magnons. The shift is defined as the integral along the magnon propagation trajectory, $\boldsymbol{l}$,

$$\Delta(\text{Im ln }\psi) = i\int_l A_\mu dl_\mu = 2i\int_{C'}\theta_a(\Psi_0)d\Psi_0^a, \tag{20}$$

where the second equality is due to Eq. (18) and $C'$ is the curve on $S^2$ formed by $\Psi_0(\boldsymbol{l})$ as the magnon travels along $\boldsymbol{l}$ in the real space. It can be shown that the integral [Eq. (20)] is zero along a geodesic. Therefore, $C'$ can be closed into a loop $C$ by a geodesic connecting the initial and final position of $\Psi_0$ [see Fig. 1(b)]. Now, $C=dV$ is a boundary of the corresponding 2D area, $V$, on $S^2$. The integral [Eq. (20)] can be further evaluated by the Stocks theorem

$$2i\int_{dV}\theta_a(\Psi_0)d\Psi_0^a = 2i\int_V \Omega_{ab}d\Psi^a\wedge d\Psi^b \tag{21}$$

with $\Omega_{ab}$ being the symplectic form from Sec. II B and $\wedge$ being the so-called wedge product. We recall now that the invariant volume on $S^2$ is $\sim\Omega_{ab}d\Psi^a\wedge d\Psi^b$, (Ref. 11) so that the integral (21) is the solid angle corresponding to $V$. This is essentially the Berry phase,[23] experienced by a magnon traveling on $S^2$ and following $\Psi_0(\boldsymbol{l})$, which in our case plays the role of the adiabatic parameter.

The instanton-induced phase shift in magnons was demonstrated numerically in Ref. 6 and it is Eq. (19) which uncovers its purely geometric origin. Note that the geometric phase is already known in magnetics, though in a different context of the phase acquired by electrons traveling through magnetic instantons.[8]

### B. Instanton background as a Bose condensate

Some of the terms in Eq. (19) have unbalanced number of "annihilation" ($\psi$) and "creation" ($\bar{\psi}$) fields, e.g., the term $\sim(\psi\partial_\mu\bar{\Psi}_0)^2$ contained in $J_\mu^2$. More terms of this kind may also come from the potential, $U$, as in Eq. (22) below. Such terms explicitly break the gauge invariance—the symmetry closely related to the particle number conservation. The physical picture is analogous to that of a Bose-condensed system. Magnons can get in and out of the condensate, $\Psi_0$, while the total number of particles is conserved. Such processes must lead to the quantum interference phenomena. For example, on passing through instantons a magnon should also experience what is known in quantum optics as squeezing[24]—a phenomenon useful for the phase-noise reduction. In fact, the squeezing is not the only application of the quantum interference in photonics. Another example is the so-called entanglement[25] and the associated teleportation problem. Similar techniques could in principle be developed for magnonics as well. Yet another point is that these couplings may also contribute to the phase shift in magnons, in addition to the Berry phase.

### C. Magnon spectrum in an instanton background

Equation (19) is a convenient tool for the systematic studies of the magnon spectrum in various nontrivial instanon backgrounds: for a given $\Psi_0$, one finds $A_\mu$, $U^{(2)}$, and diagonalizes the Gaussian part of (i.e., the first line of) Eq. (19). As an example, let us consider a DW in the local uniaxial anisotropy potential: $U\sim m_x^2+m_y^2\sim e^{-\mathcal{K}}-e^{-2\mathcal{K}}$ (the anisotropy constant can be set unity by the appropriate choice of units for energy). In this case, $\Psi_0(z)=e^{i\phi}e^z$, where $\phi$ is the azimuthal angle of $\vec{m}$ on the DW and $z$ is the out-of-plane coordinate, $A_\mu=0$ because $\Psi_0$ follows a geodesic, and the Gaussian part of the potential term in Eq. (19) is

$$(1/2)e^{2\mathcal{K}_0}U_{|ab}\psi^a\psi^b = (1-6|\Psi_0|^2e^{-2\mathcal{K}_0})|\psi|^2 - e^{-2\mathcal{K}_0}[(\bar{\Psi}_0\psi)^2 + \text{c.c.}]. \tag{22}$$

The linearized CSE to be diagonalized is

$$\omega_{n,\boldsymbol{k}_\parallel}\psi_{n,\boldsymbol{k}_\parallel}(z) = \hat{H}_1\psi_{n,\boldsymbol{k}_\parallel}(z) + \hat{H}_2\bar{\psi}_{n,-\boldsymbol{k}_\parallel}(z),$$

$$-\omega_{n,\boldsymbol{k}_\parallel}\bar{\psi}_{n,-\boldsymbol{k}_\parallel}(z) = \hat{H}_2^\dagger\psi_{n,\boldsymbol{k}_\parallel}(z) + \hat{H}_1\bar{\psi}_{n,-\boldsymbol{k}_\parallel}(z),$$

where $\hat{H}_1=-\partial_z^2+\boldsymbol{k}_\parallel^2+1-\text{sech}^2 z$, $\hat{H}_2=e^{i\phi}\,\text{sech}^2 z$, $\boldsymbol{k}_\parallel$ is the in-plane momentum, and $n$ is the quantum number of the spatial quantization in the $z$ direction. The results of the numerical diagonalization, given in Fig. 1(c), reveal the existence of the Goldstone mode (GM).[9] This is expectable, since away from the DW, where $\Psi_0=0,\ \infty$, Eq. (19) is $U(1)$ invariant, which is merely the rotational symmetry around the easy axis, while the DW breaks this invariance leading to the appearance of the GM localized on the DW. The DW GM can be viewed as the Bogoliubov sound due to the presence of the magnon condensate, i.e., of the DW. The physical essence of the DW GM is the fluctuations in $\phi$ [see Fig. 1(d)] and the fact that only one component of $\vec{m}$ is involved (unlike for magnons in which the two components of $\vec{m}$ constitute the fluctuating complex field $\psi$) can be looked upon as the squeezing of the magnon quantum between the states with the opposite (in-plane) momenta—a feature pertinent to any homogeneous (in the in-plane sense in our case) Bose-condensed system.

### D. Ginzburg-Landau perturbation series expansion

Nonlinearities experienced by magnons can be due to the interaction with other parts of the system, e.g., with phonons, or "geometrical," i.e., instanton assisted or due solely to the nonlinearity of the target space, $S^2$. At this, the geometrical nonlinearities, such as the DW-assisted three-magnon scattering given in Figs. 1(c) and 1(d), are dominant as we discuss in the next section. Equation (19) is the first systematic account of the geometric nonlinearities to all the orders and it serves as the starting point for the development of the dynamical perturbation series expansion. In particular, Eq. (19) can be used to renormalize the magnetodynamics given by Eq. (6). At this, $\psi$, must be viewed as the (dynamically or thermally) fluctuating field and the fluctuational corrections will modify the original action Eq. (6) for the instanton background, $\Psi_0$. In this line of thinking, it may be possible to get some additional insights on the Bose condensation of magnons,[26] which thus must be viewed as the (thermal or quantum)-fluctuation-induced reconfiguration of the instanton background, $\Psi_0$.

### E. Three-magnon scattering

As we just mentioned in the previous section, the geometrical nonlinearites are dominant in the magnon system. To see this, consider the lowest-order nonlinearity—the three-magnon scattering, which is known to be the most pronounced nonlinear process in ferromagnets,[10] and which as well can be phonon (or other "external" excitation) assisted. The geometrical three-magnon vertex in the first two terms of the second line of Eq. (19) is $\sim\partial_\mu\Psi_0$ while the vertex coming from the potential is

$$\frac{1}{6}e^{3\mathcal{K}_0}U_{|abc}\psi^a\psi^b\psi^c = 2e^{-2\mathcal{K}_0}(|\Psi_0|^2-1)|\psi|^2(\Psi_0\bar{\psi}+\text{c.c.}),$$

which vanishes when $\vec{m}$ is along the easy axis ($\Psi_0 \to 0,\infty$). In other words, the geometrical three-magnon scattering is instanton assisted only. This fact will not be voided by the inclusion of the effects of the external and demagnetization magnetic fields if the both fields are parallel to the easy axis. In case of strong external fields, the ferromagnet is poled, there are no instantons, and the geometrical three-magnon scattering disappears. It is also understood that the phonon-assisted three-magnon scattering is not supposed to crucially depend on the external field. It is experimentally observed[27] that in strong external fields the total three-magnon scattering weakens dramatically. Summing the above leads to the conclusion that the geometrical three-magnon scattering is indeed the strongest. This conclusion must hold as well for higher order nonlinearities, the full account of which is now enabled by Eq. (19).

### F. STT generation as a Josephson supercurrent

As we already mentioned in Sec. II C, the proposed Lagrangian formulation for magnetodynamics in terms of the complex order parameter, $\Psi$, may provide a possibly useful alternative description of STT generators.[28] Let us address here the problem of a single STT generator.

The essence of the device operation is presented in Fig. 2(a). A dc bias, $V_g$, pushes electrons from the fixed layer (FL) into the free layer (fL). The magnetizations of the fL and FL are perpendicular to each other. Upon the thermalization to the new equilibrium conditions, the arriving electrons give away their out-of-equilibrium energy and spin momentum to the magnetization of the fL. This is the "swaser" picture[29] of the STT generation.

The effective coupling between the magnetization of the fL and the nonequilibrium electrons, which arrive from the FL, is given to the lowest order in the Lagrangian density as $\mathcal{L}_{\Psi\text{-el}} \sim \vec{m}\vec{\vartheta}$, where $\vec{m}$ is the magnetization of the fL and $\vec{\vartheta}(\boldsymbol{r},t)$ is the appropriate nonequilibrium electron-density matrix. In the $\Psi$ coordinates for the fL magnetization, the effective coupling resembles that in the Josephson problem,

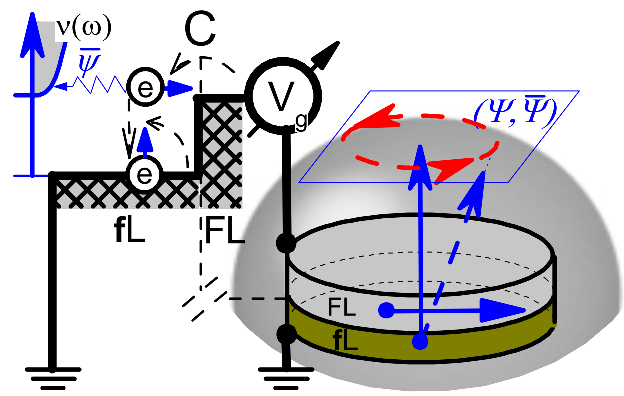
$$\mathcal{L}_{\Psi\text{-el}} = e^{-\mathcal{K}}[\vartheta(\boldsymbol{r},t)\bar{\Psi}(\boldsymbol{r},t)+\text{c.c.}]$$

with $\vartheta \sim \vartheta_x + i\vartheta_y$ being the appropriately redefined source term. In principle, $\vartheta$ can be obtained using the Keldysh nonequilibrium diagrammatic technique,[14] and its exact form depends on the physical picture adopted for the electron (spin) injection. In particular, to make the full analogy with the Josephson phenomenon, one should chose $\theta \sim e^{iV_g t}$. We do not need, however, the exact form of $\vartheta$ for our qualitative consideration.

Now, the magnetodynamics is governed by the nonhomogeneous CSE [Eq. (5)],

$$(i\partial_t + i\alpha)\Psi = \vartheta + [-D_\mu\partial_\mu + e^{-\mathcal{K}}(1-|\Psi|^2)]\Psi, \qquad (23)$$

where we phenomenologically included the damping, $\alpha$, and used again the uniaxial potential approximation. In the limit



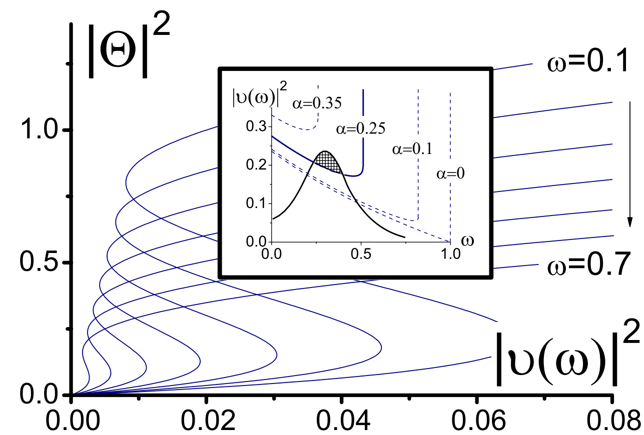




FIG. 2. (Color online) (Top) An STT stack device and its energy-band profile. The FL, separated from the fL by a contact (C), has a perpendicular magnetization (horizontal arrow) and higher electrochemical potential raised by bias voltage, $V_g$. Being pushed from the FL to fL, an electron, $e$, brings in the out-of-equilibrium energy and spin momentum (short arrow), both being transferred to the fL magnetization order parameter upon the thermalization (dashed arrows). If the energy drop, $\omega$, in the thermalization process of an electron is greater than the magnon gap, a magnon, $\bar{\psi}$, is created within its density of states $\nu(\omega)$ and under certain conditions the generation may occur. This picture of the STT generation corresponds to $\omega \to 1$ case below. (Bottom) The magnetization, $\Theta$, vs the effective pumping, $\vartheta$, as provided by Eq. (23) for $\alpha=0.1$ and for $0.1<\omega<0.7$. The upper branches of the curves correspond to the generation solutions. (Inset) The generation solutions exist if $|\vartheta(\omega)|^2$ (thick curve) is greater than $f(\omega,\alpha)$ (a set of thin curves given for several $\alpha$'s) for some range of $\omega$, as indicated by the shaded area for the case $\alpha=0.25$.

of $|\Psi| \ll 1$, this complex order parameter picture reduces to the coherent states picture of Ref. 30.

In case of the single-domain STT stack geometry, where the spatial dependence can be neglected, Eq. (23) provides a simple solution for the problem of the steady-state generation, $\Psi(t) \to \Theta e^{-i\omega t}$, $|\vartheta(\omega)|^2 = |\Theta|^2\{[e^{-\mathcal{K}}(1-|\Theta|^2)-\omega]^2+\alpha^2\}$, where $\vartheta(\omega) = \int_t \vartheta(t)e^{i\omega t}$. To answer the question of whether the generation occurs, one has to find out if for a given $\vartheta$ there exists a range of $\omega$, corresponding to the generation solutions [see Fig. 2(b)]. This criterion can be written as $|\vartheta(\omega)|^2 > f(\omega,\alpha)$, where $f$ is a complicated analytical function such that $f(\omega,\alpha)|_{\alpha\to 0} = (1/4)(1-\omega)/(1+\omega)$. Within this range of $\omega$, the sought $\omega/\Theta$ pair is the one which extremizes the GL action corresponding to Eq. (23). Note also that the rotation in the "southern" hemisphere, $|\Theta|>1$, is in the opposite temporal direction, $\omega<0$.

In general, Eq. (23), accompanied by its parental action, is more trackable than the ordinary LLE and most importantly it provides the STT generation problem with the action minimization principle. It is also interesting to note that according to the above analogy with the Josephson phenomenon, the STT generation is the ferromagnetic counterpart of the oscillating tunneling supercurrent, $\sim\sin(V_g t)$, between two superconducting order parameters externally biased by a voltage, $V_g$.

### G. Soft-mode fluctuations as a dilation field

In fact, magnons, which are the fluctuations in the orientation of the magnetization, are not the only excitations of the magnetization background. Where are also (gapped) fluctuations in the length of the magnetization known as the soft mode. These fluctuations can be incorporated into the LF by allowing $m_0$ to vary. This can be done, in turn, by the introduction of the so-called dilation field, $\Phi: m_0 \to m_0 e^{\Phi}$. In other words, this means we recall that the magnetization vector is 3D, $\vec{m}\in\mathbb{R}^3$ and make the coordinate transformation $\vec{m} \to (\Psi^a,\Phi)$, $\mathbb{R}^3\to S^2\otimes\mathbb{R}$. The invariant length of the magnetization vector takes the following form:

$$d\vec{m}^2 \sim e^{2\Phi}(d\Phi^2 + g_{ab} d\Psi^a d\Psi^b). \tag{24}$$

In terms of the dilation field, $\Phi(\boldsymbol{r},t)$, the soft-mode fluctuations are accounted for by turning to the following Lagrangian density,

$$\mathcal{L}(\Phi,\Psi^\alpha) = e^{2\Phi}\{f(\Phi)\theta_a\partial_t\Psi^a - (g_{ab}\partial_\mu\Psi^a\partial_\mu\Psi^b/2 + U_\Psi) + v_\Phi^{-2}(\partial_t\Phi)^2 - [(\partial_\mu\Phi)^2 + U_\Phi(\Phi)]\}. \tag{25}$$

Here $\theta_a$ and $g_{ab}$ are, respectively, from Eqs. (3) and (6b), $U_\Psi(\Psi^\alpha)=e^{-2\Phi}U(\Psi^\alpha)$ is the "anisotropy" potential, which relates to the previously used one as $U_\Psi(\Psi^\alpha)=e^{-2\Phi}U(\Psi^\alpha)$, $U_\Phi(\Phi)$ is the Landau part, which forces $\Phi=0$ be a ground-state value, $v_\Phi$ is a constant, which could be roughly called the "velocity" of the soft mode, and $f(\Phi)$ is some function of $\Phi$. Note that according to Eq. (24), the term $(\partial_\mu\Phi)^2$ comes from the GL functional on the same footing with the term $g_{ab}\partial_\mu\Psi^a\partial_\mu\Psi^b$. Therefore, the constant in front of it is fixed by the covariance arguments. In contrary, there is a freedom in choosing $v_\Phi$, which is thus case dependent. The same could be said about function $f(\Phi)$, which cannot be unambiguously fixed by the covariance and which must be determined from some additional physical arguments.

We believe that those must be the scaling arguments. For example, the requirement that it is only the potential $U_\Phi$, which can break the scaling symmetry, leads to the conclusion that $f=e^{-\Phi}$. With this choice of $f$, the "potential-free action" (without $U_\Phi$) is invariant under the scaling transformation: $t\to t\mu$, $\boldsymbol{r}\to\boldsymbol{r}\mu$, $\Phi\to\Phi+\log\mu$. Another possibility, which can be appropriate in case a ferromagnet is close to its Hertz-Millis ferromagnetic quantum transition,[31] is to use a separate scaling law for time as compared to space, with the relative scaling exponent typically denoted as $z\approx 3$. At this, however, in order for the potential-free action to be invariant under: $t\to t\mu^z$, $\boldsymbol{r}\to\boldsymbol{r}\mu$, $\Phi\to\Phi+\log\mu$, the initially constant $v_\Phi$ must now be a function of $\Phi$.

## V. CONCLUSION

In conclusion, we showed that the Landau-Lifshitz magnetodynamics is a member of the covariant-Schrödinger-equation family of Hamilton models. Then, using the covariant background field method we derived the Ginzburg-Landau Lagrangian formalism for magnons in an instanton background. The magnonic action turned out to be that of nonrelativistic spinless bosons coupled to the $U(1)$ gauge field and to the Bose condensate, both representing instantons. This allowed us to unveil the geometric origin of the instanton-induced phase shifts in magnons; to develop a convenient method of studying magnon spectrum in nontrivial instanton backgrounds and to demonstrate its usefulness by considering the magnonic Goldstone mode on a domain wall; to enable the Ginzburg-Landau dynamical diagrammatic technique for magnons; to reconsider the problem of the spin-transfer-torque generation as the Josephson phenomenon; and to incorporate the soft-mode fluctuations into the proposed approach.

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